Analysis 2

7 May 2024

Warm-up: If you run $114\,\mathrm{km}$ in $5\,\mathrm{hours}$, what is your "average speed"?

Speed (or velocity)

- If you run $114\,\mathrm{km}$ in $5\,\mathrm{hours}$, what is your “average speed”?

- If your position in meters after t seconds is

$$y(t) = \frac{1}{10}t^2 + 2t,$$

calculate your “average speed” (in $\mathrm{m/s}$) between $t = 2$ and $t = 10$.

Speed (or velocity)

- If your position after t seconds is

$$y(t) = \frac{1}{10}t^2 + 2t,$$

<u>estimate</u> your “*instantaneous* speed” when $t = 2$.

Speed (or velocity)

- If your position after t seconds is

$$y(t) = \frac{1}{10}t^2 + 2t,$$

calculate your "*instantaneous* speed" exactly when $t = 2$.

Analysis 1

Main topics:

- Limits
- Derivatives
  - rules for individual functions (power, trig)
  - rules for combining functions (Sum, Product, Quotient, Chain)
  - tangent lines
  - monotonicity (increasing vs. decreasing) and critical points
  - concavity (concave up vs. concave down) and inflection points
  - extrema (minima and maxima)
- Integrals

Analysis 1

Main topics:

- Limits
  - limit as $n \to \infty$ for a sequence
  - limit as $x \to \infty$ or $x \to -\infty$ for a function
  - limit as $x \to a$ for a function (a is some number)
  - limit as $x \to a^-$ or $x \to a^+$ for a function
  - graphs
  - calculations: algebra, Squeeze, L'H
- Derivatives
- Integrals

Weeks ago

For the function

$$f(x) = \frac{x^2 - x - 2}{x - 2},$$

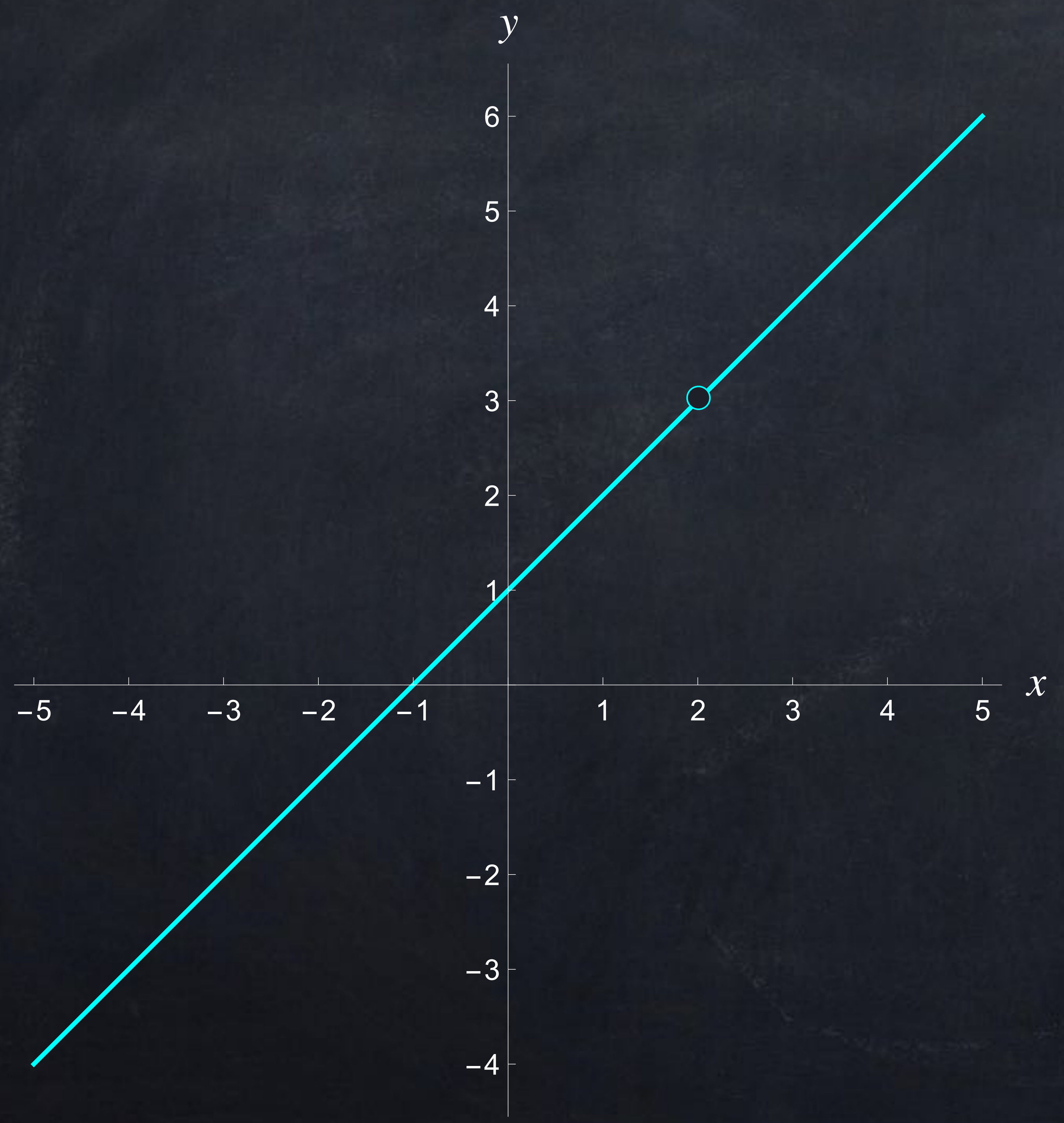
when $x = 2$,

$$f(2) = \frac{2^2 - 2 - 2}{2 - 2} = \frac{0}{0} = 🙁.$$

But if we look at the graph $y = \dfrac{x^2 - x - 2}{x - 2}$, we will be able to say more about $f(2)$.

For the function

$$f(x) = \frac{x^2 - x - 2}{x - 2},$$



All of the x-values very close to 2 give us values of $f(x)$ very close to 3.

In symbols, we write

$$\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = 3.$$

For the function

$$f(x) = \frac{x^2 - x - 2}{x - 2},$$

we can also use a table of values to find $\lim_{x \to 2} f(x)$.

x	1.8	1.9	1.99	1.999	2.001	2.005	2.1
$f(x)$	2.8	2.9	2.99	2.999	3.001	3.005	3.1

These are very close to 3.

Note: this "limit" is about what happens when the input is CLOSE to a certain value but NOT exactly equal to it. We do NOT include x = 2 in this table.

Limits as x → a

In general, we write

$$\lim_{x \to a} f(x) = L,$$

if all values of x very close a give values of $f(x)$ that are very close to L.

The equation above is said out loud as

"the limit as X goes to A of F of X equals L"

or

"the limit as X approaches A of F of X equals L".

Limits as x → a

In general, we write

$$\lim_{x \to a} f(x) = L,$$

if all values of x very close a give values of $f(x)$ that are very close to L.

There is an official definition:

- $\lim_{x \to a} f(x) = L$ means that for any $w > 0$ there exists $d > 0$ such that if $a - d < x < a + d$ and $x \neq a$ then $L - w < f(x) < L + w$.

Limits as x → a

In general, we write

$$\lim_{x \to a} f(x) = L,$$

if all values of x very close a give values of $f(x)$ that are very close to L.

There is an official definition:

- $\lim_{x \to a} f(x) = L$ means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

  if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Often, this definition is written with absolute value notation…
and with Greek letters (epsilon ε and delta δ).

Looking at "instantaneous speed" earlier, we calculated

$$\frac{\left(\frac{1}{10}(2.1)^2+4\right)-\left(\frac{1}{10}(2)^2+4\right)}{2.1-2}=2.41$$

and

$$\frac{\left(\frac{1}{10}(2.001)^2+4\right)-\left(\frac{1}{10}(2)^2+4\right)}{2.001-2}=2.4001.$$

How do we know for sure that

$$\lim_{t\to 2}\frac{\left(\frac{1}{10}t^2+2t\right)-\left(\frac{1}{10}(2)^2+4\right)}{t-2}=2.4\ ?$$

Example: find $\lim_{x \to 5} \frac{x - 5}{x^2 - 25}$.

Method 1: table

x	4.9	4.95	4.99	4.999	5.001	5.005	5.02	5.1
$f(x)$								

Method 2: graph

Method 3: algebra

Limit properties

For any numbers a and c,

- $\lim_{x \to a} c = c$ and
- $\lim_{x \to a} x = a$.

Examples: $\lim_{x \to 6}(27) = 27$ and $\lim_{x \to 6}(x) = 6$.

These should not be surprising.

Limit properties

If the limits all exist and are finite, then

- $$\lim_{x\to a}\Big(f(x)+g(x)\Big) = \Big(\lim_{x\to a} f(x)\Big) + \Big(\lim_{x\to a} g(x)\Big),$$
- $$\lim_{x\to a}\Big(f(x)\cdot g(x)\Big) = \Big(\lim_{x\to a} f(x)\Big)\Big(\lim_{x\to a} g(x)\Big),$$
- $$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$ if $\lim_{x\to a} g(x) \neq 0,$
- $$\lim_{x\to a} f\big(g(x)\big) = f\Big(\lim_{x\to a} g(x)\Big)$$ if f is a "nice" function.

with $g(x) = c$

- $$\lim_{x\to a}\Big(c\cdot f(x)\Big) = c\cdot\Big(\lim_{x\to a} f(x)\Big)$$

Later we will see exactly when $\lim_{x \to a} f\left(g(x)\right) = f\left(\lim_{x \to a} g(x)\right)$ is allowed.

For now, it is enough to know that…

- any polynomial
  - This includes x^2. → $\lim_{x \to a} \left(f(x)^2\right) = \left(\lim_{x \to a} f(x)\right)^2$
- $\sqrt[n]{x}$
- $\sin(x)$ and $\cos(x)$
- e^x and a^x with $a > 0$
- $\ln(x)$ and $\log_b(x)$ with $b > 0$

can all be used safely in this limit rule.

*You might only be allowed to use $x \geq 0$ or $x > 0$ with these functions.

Example: Calculate $\lim_{x \to 3} x^2 - 15x + 9$ using the limit properties.

$$\lim_{x \to 3} x^2 - 15x + 9 = \left(\lim_{x \to 3} x^2\right) - \left(\lim_{x \to 3} 15x\right) + \left(\lim_{x \to 3} 9\right)$$

$$= \left(\lim_{x \to 3} x\right)^2 - 15\left(\lim_{x \to 3} x\right) + \left(\lim_{x \to 3} 9\right)$$

$$= (3)^2 - 15 \cdot (3) + (9)$$

$$= -27$$

This is same as the value of $x^2 - 15x + 9$ itself when $x = 3$.

I will say more later about when we can find limits just by plugging in an x value.

Limits as $x \to \pm\infty$

The official definition

- $\lim_{x\to\infty} f(x) = L$ means that for any $\varepsilon > 0$ there exists an X such that if $x > X$ then $|f(x) - L| < \varepsilon$.

and a similar one for "$\lim_{x\to-\infty}$" can be difficult to understand at first, but these limits are very easy to see in graphs.

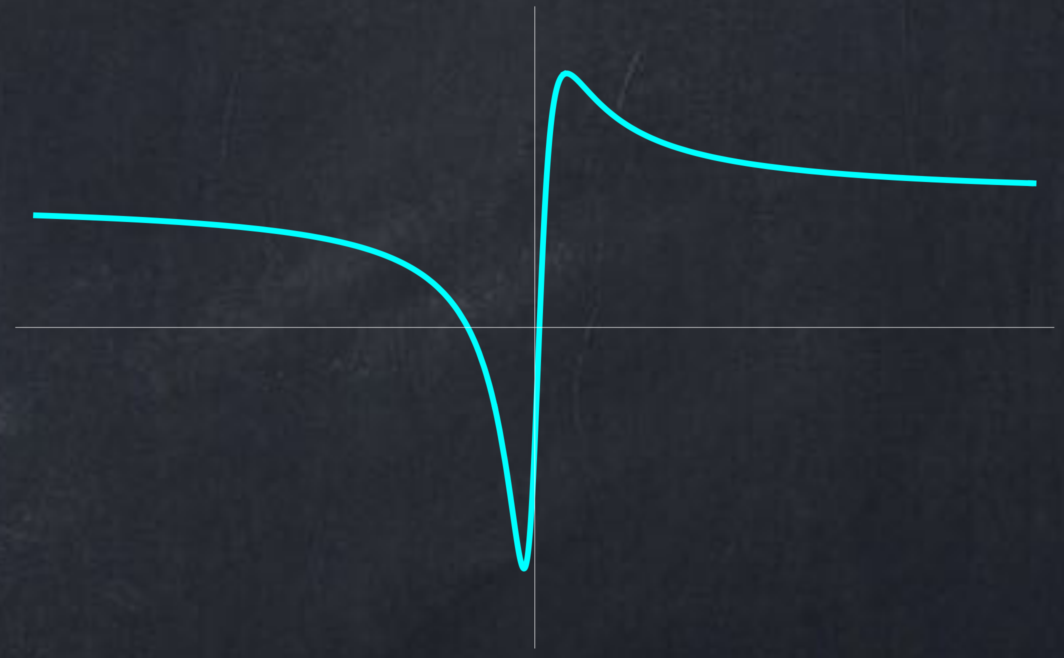
Limits as $x \to \pm\infty$

The line $y = c$ is a **horizontal asymptote** of the graph $y = f(x)$ if

$$\lim_{x \to -\infty} f(x) = c \quad \text{or} \quad \lim_{x \to \infty} f(x) = c.$$

Examples:

- $f(x) = \dfrac{8x^2 + 30x - 9}{4x^2 + 5}$ has a horizontal asymptote at $y = 2$.



- $f(x) = \dfrac{10^x}{10^x + 58}$ has a horizontal asymptote at $y = 1$ and also at $y = 0$.

It is often helpful to think of

$$\infty - 5 = \infty, \qquad \frac{\infty}{2} = \infty, \qquad \frac{14}{\infty} = 0, \qquad \infty + \infty = \infty$$

sometimes, but be careful! We *cannot* say

$$\infty - \infty = 0 \quad \text{or} \quad \frac{\infty}{\infty} = 1$$

because, for example,

$$\lim_{x \to \infty} \frac{x+1}{2x} = \frac{1}{2}, \qquad \lim_{x \to \infty} \frac{2^x}{2x+1} = \infty, \qquad \lim_{x \to \infty} \frac{\sqrt{x}}{2x+1} = 0$$

are all "$\frac{\infty}{\infty}$".

There is no way to simplify $\frac{\infty}{\infty}$ that always works.

This is an example of an **indeterminate form**. Other indeterminate forms include

$$\infty - \infty, \qquad \frac{0}{0}, \qquad 0 \times \infty, \qquad 0^0, \qquad 1^\infty, \qquad \infty^0.$$

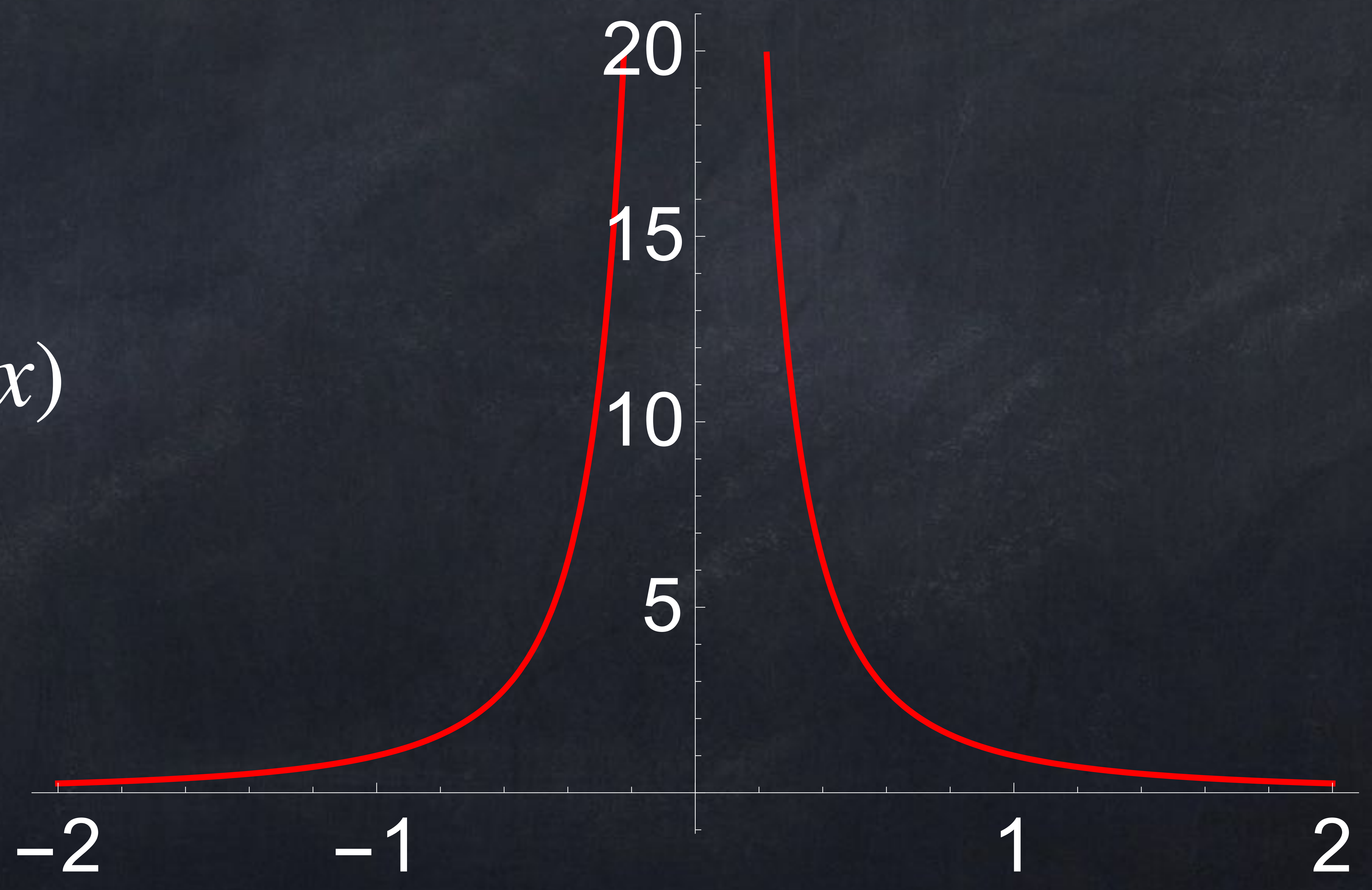
Depending on what formulas are causing 0 or $\pm\infty$ to appear, limits with these patterns can have many different values.

Infinite Limits

Sometimes the limit as x approaches some finite point will be ∞ or $-\infty$.

For example, $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

This means that for values of x very close to 0, the values of $f(x)$ are all extremely large.



Some limit properties, such as

$$\lim_{x \to a} \left(f(x) + g(x) \right) = \left(\lim_{x \to a} f(x) \right) + \left(\lim_{x \to a} g(x) \right),$$

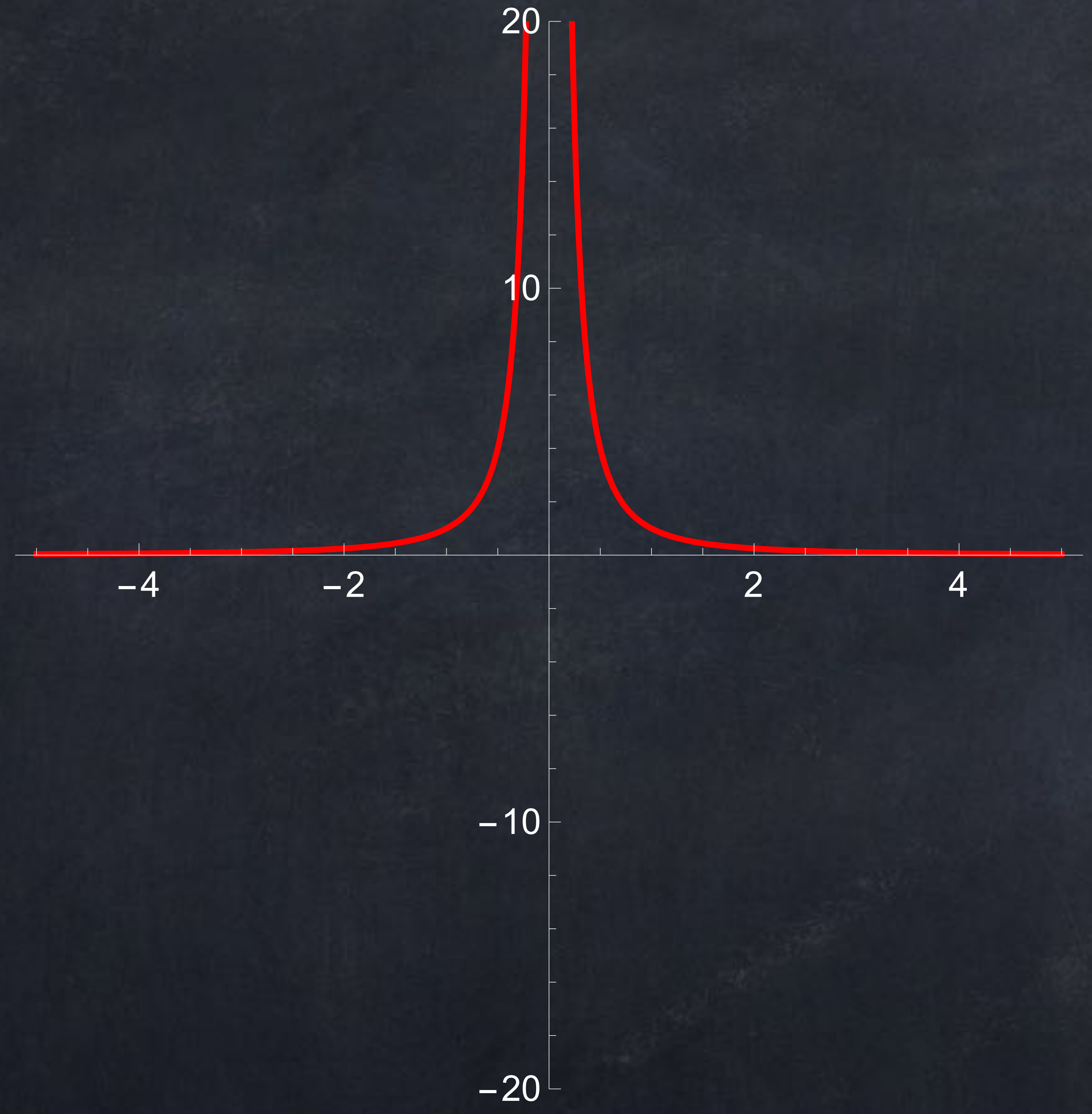
do not work with infinite limits.

Patterns like $\infty - \infty$ and $\frac{\infty}{\infty}$ are indeterminate forms. We can ***not*** just say that "$\infty - \infty = 0$" because subtracting functions with infinite limits can give many different answers. Both

- $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{x^2} \right) = 0$
- $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{x^4} \right) = -\infty$

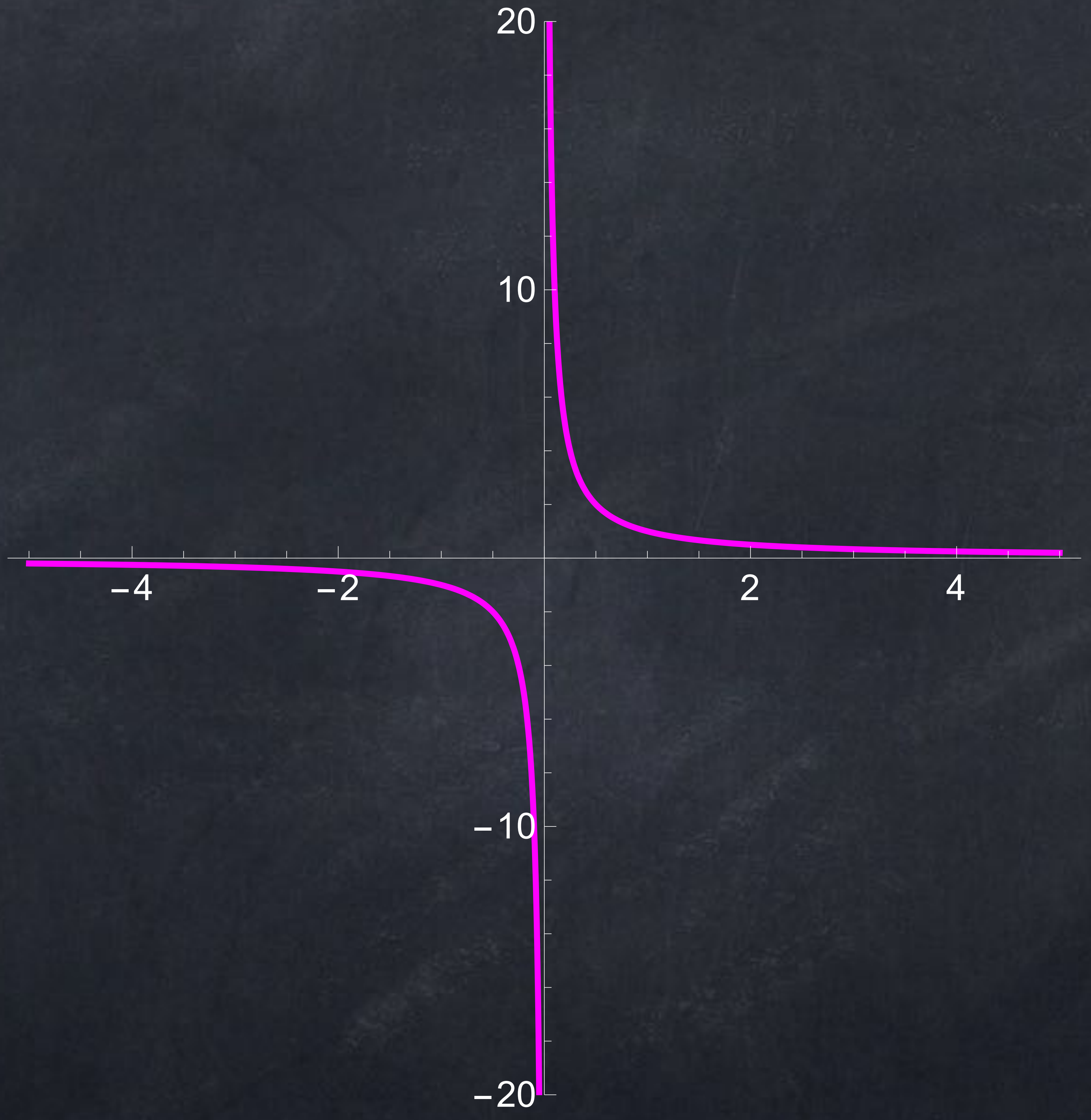
are "$\infty - \infty$" in some way.

$$y = \frac{1}{x^2}$$



$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

$$y = \frac{1}{x}$$



$$\lim_{x \to 0} \frac{1}{x} \text{ doesn't exist}$$

One-sided Limits

We write

$$\lim_{x \to a^-} f(x)$$

for the "**limit as x approaches a from the left**" or "… from below". This means we only look at x values that are less than a.

Similarly,

$$\lim_{x \to a^+} f(x)$$

means the "**limit as x approaches a from the right**" or "… from above", where we only look at x values that are more than a.

One-sided Limits

We write

$$\lim_{x \to a^-} f(x)$$

for the “limit as x approaches a from the left.” This means we only look at x values that are less than a.

Example: $\lim_{x \to 0^-} x\sqrt{1+\frac{1}{x^2}}$

One-sided Limits

Note: writing

$$4^+$$

by itself does not mean anything (like $\sqrt{}$ or $|\ \ |$ alone). This should only be written as part of a limit:

$$\lim_{x \to 4^+} f(x).$$

Some books use $\lim_{x \nearrow 4} f(x)$ and $\lim_{x \searrow 4} f(x)$ instead of $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$.

One-sided Limits

All of the limit rules for functions, such as

- $\lim_{x\to a}\left(f(x)+g(x)\right) = \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a} g(x)\right),$

can also be used with one-sides limits:

- $\lim_{x\to a^-}\left(f(x)+g(x)\right) = \left(\lim_{x\to a^-} f(x)\right) + \left(\lim_{x\to a^-} g(x)\right),$
- $\lim_{x\to a^+}\left(f(x)+g(x)\right) = \left(\lim_{x\to a^+} f(x)\right) + \left(\lim_{x\to a^+} g(x)\right).$

One-sided limits are related to standard limits in the following way:

> If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ have different values, or if at least one of them does not exist, then $\lim_{x\to a} f(x)$ does not exist.

Logically, this also means that

- if $\lim_{x\to a} f(x)$ exists then $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.